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Exact form factors for the Josephson tunneling current and relative particle number fluctuations in a model of two coupled Bose-Einstein condensates

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Abstract

Form factors are derived for a model describing the coherent Josephson tunneling between two coupled Bose-Einstein condensates. This is achieved by studying the exact solution of the model in the framework of the algebraic Bethe ansatz. In this approach the form factors are expressed through determinant representations which are functions of the roots of the Bethe ansatz equations.

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The experimental realisation of Bose-Einstein condensates in atomic alkali gases is generating a great deal of theoretical activity in order to understand the nature of coherent Josephson tunneling between coupled systems. A simple two-mode Hamiltonian which has been widely studied (see [1] for a review) takes the form

$$H = \frac{1}{8}K\hat{n}^2 - \frac{\Delta\mu}{2}\hat{n} - \frac{\mathcal{E}_\mathcal{J}}{2}(a_1^\dagger a_2 + a_2^\dagger a_1). \tag{1}$$

where $a_1^\dagger, a_2^\dagger$ denote the single-particle boson creation operators, $\hat{N}_1 = a_1^\dagger a_1$, $\hat{N}_2 = a_2^\dagger a_2$ are number operators and $\hat{n} = \hat{N}_1 - \hat{N}_2$ is the relative particle number operator. Note that the total number operator $\hat{N} = \hat{N}_1 + \hat{N}_2$ provides a good quantum number since $[H, \hat{N}] = 0$.

While this model has been studied using a variety of techniques such as Gross-Pitaevskii states [1], mean-field theory [2], quantum phase model [3] and an exact quantum phase model [4], it also admits an exact solution in the framework of the algebraic Bethe ansatz, given in [5] in the guise of the "discrete self-trapping dimer" model, which has been largely unexplored. In [6], we have used the exact solution to determine the asymptotic behaviour of the energy spectrum in the Fock ($N << K/\mathcal{E}_J$) and Rabi ($N^{-1} >> K/\mathcal{E}_\mathcal{J}$) regions. It was also shown that asymptotic thermodynamic properties can be deduced for the Fock region at low temperature and the Rabi region for all temperatures. Here, we will limit ourselves to the case $\Delta\mu = 0$ and continue our analysis of the Bethe ansatz solution to yield explicit exact form factors for the Josephson tunneling current

$$\mathcal{J} = i(a_1^\dagger a_2 - a_2^\dagger a_1), \tag{2}$$

as well as the relative particle number $\hat{n}$ and $\hat{n}^2$, which are applicable for all couplings. This opens an avenue to explore the behaviour of the Hamiltonian in the cross over between the Fock, Rabi and Josephson ($N^{-1} << K/\mathcal{E}_J << N$) regions. Recall that in Josephson's original work on macroscopic superconductors [7], the tunneling current is a manifestation of the relative phase of the wave functions of two superconductors separated by an insulating barrier. As phase and particle number are canonically conjugate quantum variables, the quantum fluctuations of the relative particle number are of primary importance in understanding the physics in the present model, given that there are technical difficulties which prevent a simple definition for the phase variable [1, 8]. Our results for the form factors provide

an initial step towards the calculation of these fluctuations. The method that we will adopt follows that proposed for the form factors of the Bose gas with delta-function interactions and the one-dimensional Heisenberg model [9, 10]. This yields determinant representations for the form factors which are functions of the roots of the Bethe ansatz equations that arise from the exact solution. Furthermore, it is straightforward to express these results in a time-dependent form.

First we will review the basic features of the exact solution of (1) via the algebraic Bethe ansatz, as discussed in [5]. The theory of exactly solvable quantum systems in this setting relies on the existence of a solution of the Yang-Baxter equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v). \tag{3}$$

Here $R_{jk}(u)$ denotes the matrix on $V \otimes V \otimes V$ acting on the j-th and k-th spaces and as the identity on the remaining space. The R-matrix solution may be viewed as the structural constants for the Yang-Baxter algebra generated by the monodromy matrix $T(u)$

$$R_{12}(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u-v). \tag{4}$$

For a given R-matrix, there are a variety of realisations of the Yang-Baxter algebra. For the $su(2)$ invariant R-matrix,

$$R(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5}$$

with the rational functions $b(u) = u/(u+\eta)$ and $c(u) = \eta/(u+\eta)$, there is a realisation of the Yang-Baxter algebra in terms of canonical boson operators which reads [11]

$$L(u) = \begin{pmatrix} u + \eta\hat{N} & a \\ a^\dagger & \eta^{-1} \end{pmatrix},$$

such that

$$R_{12}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u-v).$$

The co-multiplication behind the Yang-Baxter algebra allows us to choose the monodromy matrix

$$T(u) \quad = \quad L_1(u+\omega)L_2(u-\omega)$$

$$= \begin{pmatrix} (u+\omega+\eta\hat{N}_1)(u-\omega+\eta\hat{N}_2)+a_2^\dagger a_1 & (u+\omega+\eta\hat{N}_1)a_2+\eta^{-1}a_1 \\ (u-\omega+\eta\hat{N}_2)a_1^\dagger+\eta^{-1}a_2^\dagger & a_1^\dagger a_2+\eta^{-2} \end{pmatrix} \tag{6}$$

Defining the transfer matrix through $t(u) = \mathrm{tr}\,(T(u))$, it follows from (3) that the transfer matrices commute for different values of the spectral parameter; viz.

$$[t(u),\, t(v)] = 0 \quad \forall u, v.$$

In the present case, we have explicitly

$$t(u) \;=\; u^2 + u\eta\hat{N} + \eta^2\hat{N}_1\hat{N}_2 + \eta\omega\hat{n} + a_2^\dagger a_1 + a_1^\dagger a_2 + \eta^{-2} - \omega^2.$$

Then

$$t'(0) = \left.\frac{dt}{du}\right|_{u=0} = \eta\hat{N}$$

and it is easy to verify that the Hamiltonian is related with the transfer matrix $t(u)$ by

$$H = -\kappa\left(t(u) - \frac{1}{4}(t'(0))^2 - ut'(0) - \eta^{-2} + \omega^2 - u^2\right),$$

where the following identification has been made for the coupling constants

$$\begin{aligned} \frac{K}{4} &= \frac{\kappa\eta^2}{2}, \\ \frac{\Delta\mu}{2} &= -\kappa\eta\omega, \\ \frac{\mathcal{E}_{\mathcal{J}}}{2} &= \kappa. \end{aligned}$$

The solution of (1) via the algebraic Bethe ansatz is obtained by utilizing the commutation relations of the Yang-Baxter algebra. Setting

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \tag{7}$$

we have from the defining relations (4) that

$$\begin{aligned} &[A(u), A(v)] = [D(u), D(v)] = 0, \\ &[B(u), B(v)] = [C(u), C(v)] = 0, \\ &A(u)C(v) = \frac{u-v+\eta}{u-v}C(v)A(u) - \frac{\eta}{u-v}C(u)A(v), \\ &D(u)C(v) = \frac{u-v-\eta}{u-v}C(v)D(u) + \frac{\eta}{u-v}C(u)D(v). \end{aligned} \tag{8}$$

An explicit representation of (7) is obtained from (6) with the identification

$$
\begin{aligned}
A(u) &= (u+\omega+\eta N_1)(u-\omega+\eta N_2)+a_2^\dagger a_1 \\
B(u) &= (u+\omega+\eta N_1)a_2+\eta^{-1}a_1 \\
C(u) &= (u-\omega+\eta N_2)a_1^\dagger+\eta^{-1}a_2^\dagger \\
D(u) &= a_1^\dagger a_2+\eta^{-2}.
\end{aligned}
$$

Choosing the state $|0\rangle = |0\rangle_1 |0\rangle_2$ as the pseudovacuum, the eigenvalues $a(u)$ and $d(u)$ of $A(u)$ and $D(u)$ on $|0\rangle$ are

$$
\begin{aligned}
a(u) &= (u+\omega)(u-\omega), \\
d(u) &= \eta^{-2}.
\end{aligned}
$$

Next choose the Bethe state

$$
|\vec{v}\rangle \equiv |v_1, ..., v_N\rangle = \prod_{i=1}^{N} C(v_i)\,|0\rangle\,. \tag{9}
$$

Note that because $[C(u),\, C(v)] = 0$, the ordering is not important in (9). It is also clear that these states are eigenstates of $\hat{N}$ with eigenvalue N. The approach of the algebraic Bethe ansatz is to use the relations (8) to determine the action of $t(u)$ on $|\vec{v}\rangle$. The result is

$$
\begin{aligned}
t(u)\,|\vec{v}\rangle &= \Lambda(u,\,\vec{v})\,|\vec{v}\rangle \\
&\quad - \left(\sum_{i}^{N}\frac{\eta a(v_i)}{u-v_i}\prod_{j\neq i}^{N}\frac{v_i-v_j+\eta}{v_i-v_j}\right)|v_1, ...v_{i-1}, u, v_{i+1}, ..., v_N\rangle \\
&\quad + \left(\sum_{\alpha}^{N}\frac{\eta d(v_i)}{u-v_i}\prod_{j\neq i}^{N}\frac{v_i-v_j-\eta}{v_i-v_j}\right)|v_1, ...v_{i-1}, u, v_{i+1}, ..., v_N\rangle
\end{aligned} \tag{10}
$$

where

$$
\Lambda(u,\,\vec{v}) = a(u)\prod_{i=1}^{N}\frac{u-v_i+\eta}{u-v_i} + d(u)\prod_{i=1}^{N}\frac{u-v_i-\eta}{u-v_i}. \tag{11}
$$

The above shows that $|\vec{v}\rangle$ becomes an eigenstate of the transfer matrix with eigenvalue (11) whenever the Bethe ansatz equations (BAE)

$$
\eta^2(v_i^2-\omega^2) = \prod_{j\neq i}^{N}\frac{v_i-v_j-\eta}{v_i-v_j+\eta} \tag{12}
$$

are satisfied. As N is the total number of bosons, we expect $N+1$ solutions of the BAE. Note that in the derivation of the BAE it is required that $v_i \neq v_j \,\forall\, i,\, j$. For example, the solution

$$v_j = \pm(i)^{(N+1)}\sqrt{\eta^{-2}+\omega^2}, \qquad \forall j$$

of (12) is invalid, except when $N=1$. This is a result of the Pauli Principle for Bethe ansatz solvable models as developed in [12] for the Bose gas. We will not reproduce the proof for the present case, as it follows essentially the same lines as [12]. For a given valid solution of the BAE, the energy of the Hamiltonian is obtained from the transfer matrix eigenvalues (11) and reads

$$E(\vec{v}) = -\kappa\left(\eta^{-2}\prod_{j}^{N}\eta^2(v_j-\omega+\eta)(v_j+\omega) - \frac{\eta^2 N^2}{4} - \eta\omega N - \eta^{-2}\right). \quad (13)$$

A consequence of the above construction is that form factors, and consequently correlation functions, can be computed for this model. The method we employ follows that used for the Bose gas and one-dimenisonal Heisenberg chain [9, 10]. The reason we can adopt this approach is because the solution of these models and (1) are based on the same R-matrix (5). Below we will restrict our attention to $\omega = 0$ (for reasons which will become clear) and give explicit results for $\hat{n}$, $\hat{n}^2$ and $\mathcal{J}$.

A curiosity of this model is that the representation of the Yang-Baxter algebra is non-unitary; viz.

$$C^\dagger(u) \neq B(u).$$

In the case when $\omega = 0$ it is, however, equivalent to a unitary representation since in this instance we have

$$C^\dagger(u) = P.B(u).P$$

where P is the permutation operator. The permutation operator is defined by the action on the Fock basis

$$P.(a_1^\dagger)^j(a_2^\dagger)^k\left|0\right\rangle = (a_1^\dagger)^k(a_2^\dagger)^j\left|0\right\rangle.$$

Note that for $\omega = 0$ (which will be assumed hereafter), we have

$$[P,\, H] = 0$$

which means that the energy eigenstates are also eigenstates of P. Moreover, $P^2 = I$ shows that P has eigenvalues ± 1.

There exists a formula due to Slavnov [13] for the scalar product of states obtained via the algebraic Bethe ansatz for the R-matrix (5), which when applied to this model is

$$\begin{aligned} S(\vec{v} : \vec{u}) &= \langle 0| B(v_1)...B(v_N) C(u_1)...C(u_N) |0\rangle \\ &= \frac{\det F(\vec{u} : \vec{v})}{\prod_{j>k}(u_k - u_j) \prod_{\alpha<\beta}(v_\beta - v_\alpha)} \end{aligned}$$

where

$$F_{ab} = \frac{\eta^{-1}}{u_b - v_a} \left(u_b^2 \prod_{m \neq a}^{N} (v_m - u_b - \eta) - \eta^{-2} \prod_{m \neq a}^{N} (v_m - u_b + \eta) \right),$$

the parameters $\{v_\alpha\}$ satisfy the Bethe ansatz equations (12) and $\{u_j\}$ are arbitrary. Note that when $|\vec{u}\rangle = |\vec{v}\rangle$ we need to take a limit for the diagonal entries which gives

$$\begin{aligned} F_{aa} &= -2\eta^{-2} v_a \prod_{l}^{N} (v_l - v_a - \eta) - \eta^{-4} \prod_{l}^{N} (v_l - v_a + \eta) \sum_{m \neq a}^{N} \frac{1}{v_m - v_a + \eta} \\ &\quad + \prod_{l}^{N} (v_l - v_a - \eta) \sum_{m \neq a}^{N} \frac{\eta^{-2} v_a^2}{v_m - v_a - \eta}. \end{aligned}$$

Consider

$$\begin{aligned} S(\vec{v} : \vec{v}) &= \langle 0| B(v_1)...B(v_N) C(v_1)...C(v_N) |0\rangle \\ &= \left\langle 0| P C^\dagger(v_1)...C^\dagger(v_N) P C(v_1)...C(v_N) |0 \right\rangle \\ &= \epsilon(\vec{v}) \langle \vec{v} | \vec{v} \rangle \end{aligned}$$

where $\epsilon(\vec{v}) = \pm 1$ is the eigenvalue of P; viz.

$$P |\vec{v}\rangle = \epsilon(\vec{v}) |\vec{v}\rangle .$$

This quantity can be determined to be given by

$$\epsilon(\vec{v}) = \prod_{j}^{N} \eta v_j.$$

Hence, from the Slavnov formula, the norms of the eigenstates

$$\begin{aligned} ||\vec{v}|| &= \langle \vec{v}|\vec{v}\rangle^{1/2} \\ &= |S(\vec{v}:\vec{v})|^{1/2} \end{aligned}$$

are obtained directly.

Let us define

$$\chi = A(0) - D(0) = \eta^2 N_1 N_2 + i\mathcal{J} - \eta^{-2}$$

where $\mathcal{J}$ is defined by (2). In analogy with (10) we find

$$\begin{aligned} &\chi\,|\vec{u}\rangle \\ &= \theta(\vec{u})\,|\vec{u}\rangle \\ &+ \sum_i^N \frac{\eta a(u_i)}{u_i} \prod_{j\neq i}^N \frac{u_i - u_j + \eta}{u_i - u_j}\,|u_1, ...u_{i-1}, 0, u_{i+1}, ..., u_N\rangle \\ &+ \sum_i^N \frac{\eta d(u_i)}{u_i} \prod_{j\neq i}^N \frac{u_i - u_j - \eta}{u_i - u_j}\,|u_1, ...u_{i-1}, 0, u_{i+1}, ..., u_N\rangle \end{aligned}$$

where

$$\begin{aligned} \theta(\vec{u}) &= a(0)\prod_{i=1}^N \frac{u_i - \eta}{u_i} - d(0)\prod_{i=1}^N \frac{u_i + \eta}{u_i} \\ &= -\eta^{-2}\prod_{i=1}^N \frac{u_i + \eta}{u_i}. \end{aligned}$$

Using the Slavnov formula we can calculate the form factors for χ. Suppose that $|\vec{v}\rangle$ is an eigenstate of the Hamiltonian. Then

$$\begin{aligned} &\langle \vec{v}|\chi|\vec{u}\rangle \\ &= \theta(\vec{u})\epsilon(\vec{v})S(\vec{v}:\ \vec{u}) \\ &+ \sum_i^N \frac{\eta a(u_i)}{u_i} \prod_{j\neq i}^N \frac{u_i - u_j + \eta}{u_i - u_j}\epsilon(\vec{v})S(\vec{v}: u_1, ...u_{i-1}, 0, u_{i+1}, ..., u_N) \\ &+ \sum_\alpha^N \frac{\eta d(u_i)}{u_i} \prod_{j\neq i}^N \frac{u_i - u_j - \eta}{u_i - u_j}\epsilon(\vec{v})S(\vec{v}:\ u_1, ...u_{i-1}, 0, u_{i+1}, ..., u_N). \end{aligned}$$

Now we let $|\vec{u}\rangle$ be an eigenstate of the Hamiltonian (1). The above formula can be simplified considerably (cf. [10])

$$\begin{aligned}&\langle\vec{v}|\chi|\vec{u}\rangle\\ &\quad=\frac{-\eta^{N-2}\prod_\alpha(v_\alpha+\eta)}{\prod_{j>k}(u_k-u_j)\prod_{\alpha<\beta}(v_\beta-v_\alpha)}\det\Big(F(\vec{v}:\,\vec{u})-2\eta^{-3}\epsilon(\vec{u})\epsilon(\vec{v})Q(\vec{v}:\,\vec{u})\Big)\end{aligned}\tag{14}$$

where $Q(\vec{v}:\,\vec{u})$ is a rank one matrix with entries

$$Q_{ab}=\frac{\prod_j(u_j-u_b+\eta)}{v_a(v_a+\eta)}.$$

This is our main result. We remark that because the basis states are also Hamiltonian eigenstates, it is straightforward to write down the time-dependent form factors

$$\langle\vec{v}|\chi(t)|\vec{u}\rangle=\exp(-it\,(E(\vec{v})-E(\vec{u})))\,\langle\vec{v}|\chi|\vec{u}\rangle\tag{15}$$

where the energies $E(\vec{v})$ are given by (13). Remarkably, from eqs. (14, 15) all the time-dependent form factors for $\hat{n}$, $\hat{n}^2$ and $\mathcal{J}$ can be obtained. This is achieved by exploiting the symmetry of the Hamiltonian under P, which we will now explain.

The following corollary is easily proved. If $\epsilon(\vec{v})\neq\epsilon(\vec{u})$ then

$$\left\langle\vec{v}|\hat{N}_1\hat{N}_2|\vec{u}\right\rangle=0.$$

If $\epsilon(\vec{v})=\epsilon(\vec{u})$ then

$$\langle\vec{v}|\mathcal{J}|\vec{u}\rangle=0.$$

The result follows from the observation

$$\begin{aligned}P\hat{N}_1\hat{N}_2&=\hat{N}_1\hat{N}_2P,\\ P\mathcal{J}&=-\mathcal{J}P.\end{aligned}$$

We now find that

$$\left\langle\vec{v}|\hat{N}_1\hat{N}_2|\vec{u}\right\rangle=\eta^{-2}\,\langle\vec{v}|\chi(t)|\vec{u}\rangle+\eta^{-4}\,\langle\vec{v}|\vec{u}\rangle$$

if $\epsilon(\vec{v})=\epsilon(\vec{u})$, and is zero otherwise. Also

$$\langle\vec{v}|\mathcal{J}|\vec{u}\rangle=-i\,\langle\vec{v}|\chi(t)|\vec{u}\rangle$$

if $\epsilon(\vec{v}) \neq \epsilon(\vec{u})$, and is zero otherwise.

The above shows that the form factors for $\mathcal{J}$ are obtained directly from those of $\chi(t)$. Those for $\hat{n}^2$ also follow, since we have $\hat{n}^2 = \hat{N}^2 - 4\hat{N}_1\hat{N}_2$ and the states (9) are automatically eigenstates of $\hat{N}$ with eigenvalue N. Thus

$$\left\langle \vec{v}|\hat{n}^2|\vec{u}\right\rangle = N^2 \left\langle \vec{v}|\vec{u}\right\rangle - 4\left\langle \vec{v}|\hat{N}_1\hat{N}_2|\vec{u}\right\rangle .$$

To obtain the form factors for $\hat{n}$, we use the fact that $\mathcal{J}$ is the time derivative of $\hat{n}$, so

$$\mathcal{J} = \frac{i}{\kappa}[\hat{n},\, H]$$

which gives

$$\left\langle \vec{v}|\hat{n}|\vec{u}\right\rangle = \frac{i\kappa}{E(\vec{v}) - E(\vec{u})} \left\langle \vec{v}|\mathcal{J}|\vec{u}\right\rangle .$$

In principle, the correlation functions

$$\left\langle \theta \right\rangle_{\Psi} = \frac{\left\langle \Psi|\theta|\Psi\right\rangle}{\left\langle \Psi|\Psi\right\rangle}$$

where $\theta = \hat{n}$, $\hat{n}^2$ or $\mathcal{J}$, and $|\Psi\rangle$ is an arbitrary state, can be expressed in terms of the form factors through completeness relations. In particular, for a given $|\Psi\rangle$ the quantum fluctuations of the relative number operator

$$\delta(\Psi;\hat{n}) = \left\langle \hat{n}^2 \right\rangle_{\Psi} - \left\langle \hat{n}\ \right\rangle_{\Psi}^2$$

can be computed from these results.

In conclusion, we have shown that the algebraic Bethe ansatz solution of (1) provides a means to calculate form factors, and in turn correlation functions, for the model. We have yielded explicit results for the case $\Delta\mu = 0$. An outstanding task is to extend these results to the case $\Delta\mu \neq 0$, which presents a challenging problem because of the non-unitary representation of the Yang-Baxter algebra in this instance, and the breaking of symmetry under P.

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